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On Generation of Pythagorean triples

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Abstract

The problem discussed here is the determination of integer values of the generators m,n (m>n>0) in the pythagorean triple $(m^2-n^2, 2mn, m^2+n^2)$ and that of two arbitrary non-zero positive parameters k ,t such that the relation

$(a+k)^2+(b+t)^2=(c+k)^2$ holds ,where a ,b represent the legs and c ,the hypotenuse of the above mentioned Pythagorean triple.

Key words: Pythagorean triple , Generation of Pythagorean triple

Notations

$$t_{m,n}=n[1+\frac{(n-1)(m-2)}{2}]$$

$$P_n^3=\frac{n(n+1)(n+2)}{6}$$

$$P_n^4=\frac{n(n+1)(2n+1)}{6}$$

$$P_n^5=\frac{n^2(n+1)}{2}$$

Introduction

One of the basic theorems of elementary geometry is the Pythagorean theorem and has been studied by many mathematicians since antiquity. The study of Pythagorean triangles is also connected with the solution in integers of the ternary homogeneous quadratic equation $x^2+y^2=z^2$.In fact , Pythagorean triangle is a treasure house in which the search for hidden connections is a treasure hunt. For varieties of fascinating problems , one may refer [1-9]. This communication aims in the determination of integer values of the generators m,n (m>n>0) in the pythagorean triple $(m^2-n^2, 2mn, m^2+n^2)$ and that of two arbitrary non-zero positive parameters k ,t such that the relation $(a+k)^2+(b+t)^2=(c+k)^2$ holds, where a ,b represent the legs and c ,the hypotenuse of the above mentioned Pythagorean triple.

Methodology

Let a,b be the legs and c be the hypotenuse of Pythagorean triangle T (a,b,c) whose generators are m,n (m>n>0). Let k,t be two non-zero distinct positive integers such that

$$(a+k)^2+(b+t)^2=(c+k)^2 \quad (1)$$

Taking the values of a, b, c in (1) as

$$a=m^2-n^2, b=2mn, c=m^2+n^2 \quad (2)$$

and treating (1) as quadratic equation in t , we have

$$\begin{aligned} t &= -b+\sqrt{b^2+2k(c-a)} \\ &= -2mn+\sqrt{4m^2n^2+4kn^2} \\ &= 2n[-m+\sqrt{m^2+k}\,] \end{aligned} \quad (3)$$

It is to be noted that the negative sign before the square-root in (3) is not considered as we require t>0.The square-root in (3) is eliminated when

$$m=s+1, k=n^2+2n+2ns \quad (4)$$

Substituting (4) in (3) , we have

$$t=2n^2 \quad (5)$$

Thus , employing the values of a, b, c, k and t in (1) ,the required generated Pythagorean triple is given by

$$(m^2+2nm, 2n(m+n), m^2+2n(m+n)) \quad (6)$$

Let A(m,n) and P(m,n) denote respectively the area and perimeter of the Pythagorean triangle (6). Then , we have

$$P(m,n)=2(m+n)(m+2n)$$
$$A(m,n)=mn(m+n)(m+2n)$$

Denoting the triple (6) by (A_0, B_0, C_0),we observe the following :

$$\begin{aligned} 2P-A_0 &= 3m^2+10mn+8n^2 \\ 2P-B_0 &= 4m^2+10mn+6n^2 \\ 2P+C_0 &= 5m^2+14mn+10n^2 \end{aligned} \qquad (7)$$

It is worth to mention that the triple $(2P-A_0, 2P-B_0, 2P+C_0)$ is a Pythagorean triple

A few interesting observations in connection with the area and perimeter of the pythagorean triangle given by (6) are presented below:

1. $P(7n,n)$ is a perfect square
2. Each of the following expressions is a square multiple of 6
   $P(2n,n)$,$2P(23n,n)$, $P(47n,n)$
3. $P(m,1)=4t_{3,m+1}$
4. $\frac{A(n+1,n)}{P(n+1,n)}=t_{3,n}$
5. $\frac{A((4n-2),n)}{P((4n-2),n)}=t_{6,n}$
6. $\frac{A(n(n+1),n)}{P(n(n+1),n)}=P_n^5$
7. $\frac{A((n+1)(n+2),n)}{P((n+1)(n+2),n)}=3P_n^3$
8. $\frac{A((n+1)(2n+1),n)}{P((n+1)(2n+1),n)}=3P_n^4$
9. $P(m+2,n)-2P(m+1,n)+P(m,n)=4$

Note 1

Let M be a third order matrix given by

$$M = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix} \quad (8)$$

Take

$$A_1 = 2P - A_0, B_1 = 2P - B_0, C_1 = 2P - C_0$$

Express (7) in the matrix form as

$$(A_1, B_1, C_1) = M * (A_0, B_0, C_0)^t$$

where t is the transpose and M is given by (8). The repetition of the above process leads to the Pythagorean triple in general (A_n, B_n, C_n) as presented below:

$$(A_n, B_n, C_n) = M^n * (A_0, B_0, C_0)^t$$

where

$$M^n = \begin{pmatrix} \frac{Y_{n-1} + (-1)^n}{2} & \frac{Y_{n-1} - (-1)^n}{2} & X_{n-1} \\ \frac{Y_{n-1} - (-1)^n}{2} & \frac{Y_{n-1} + (-1)^n}{2} & X_{n-1} \\ X_{n-1} & X_{n-1} & Y_{n-1} \end{pmatrix}$$

in which

$$Y_{n-1}^2 = 2X_{n-1}^2 + 1, n = 1,2,3,....$$

Note 2

Taking the values of a, b, c in (1) as

$$b = m^2 - n^2, a = 2mn, c = m^2 + n^2 \quad (9)$$

and treating (1) as quadratic equation in t , we have

$$t = -b + \sqrt{b^2 + 2k(c-a)}$$
$$= -(m^2 - n^2) + \sqrt{(m^2 - n^2)^2 + 2k(m-n)^2} \qquad (10)$$
$$= -(m^2 - n^2) + (m-n)\sqrt{(m+n)^2 + 2k}$$

It is to be noted that the negative sign before the square-root in (10) is not considered as we require t>0.The square-root in (10) is eliminated when

$$m + n = s + 2, k = 2n^2 + 4n + 2ns \qquad (11)$$

Substituting (11) in (10) , we have

$$t = 2n(m-n) \qquad (12)$$

Thus , employing the values of a, b, c, k and t from (9),(11)&(12) in (1),

the required generated Pythagorean triple is given by

$$(4n^2 + 4nm, (m-n)(m+3n), m^2 + n(2m+5n))$$

Conclusion

In this paper ,the aim is the determination of integer values of the generators m,n (m>n>0) in the pythagorean triple $(m^2 - n^2, 2mn, m^2 + n^2)$ and that of two arbitrary non-zero positive parameters k ,t such that the relation

$(a+k)^2 + (b+t)^2 = (c+k)^2$ holds ,where a ,b represent the legs and c ,the hypotenuse of the above mentioned Pythagorean triple. One may search for Pythagorean triangles with other suitable characterizations.

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